

A remark on the single scattering preconditioner applied to boundary integral equations*

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Abstract

This article deals with boundary integral equation preconditioning for the multiple scattering problem. The focus is put on the single scattering preconditioner, corresponding to the diagonal part of the integral operator, for which two results are proved. Indeed, after applying this geometric preconditioner, it appears that, firstly, every direct integral equations become identical to each other, and secondly, that the indirect integral equation of Brakhage-Werner becomes equal to the direct integral equations, up to a change of basis. These properties imply in particular that the convergence rate of a Krylov subspaces solver will be exactly the same for every preconditioned integral equations. To illustrate this, some numerical simulations are provided at the end of the paper.

1 Introduction

This paper is devoted to the numerical resolution of acoustic multiple scattering problems in the time-harmonic regime. Here, multiple scattering means that the medium contains more than one obstacle contrary to single scattering where only one scatterer is considered. Multiple scattering arises in many applications and problems such as wave propagation in photonics crystals or trabecular bones modeling. Among all the numerical methods to solve scattering problems, the focus is put here on boundary integral equations. These approaches, based on the integral representation of the scattered field, reduce the initial boundary value problem to an integral equation on the surface of the obstacles, decreasing the dimension by one. However, after discretization, the non-localness of the integral operator leads to a full matrix. Moreover, the mesh of the domain must be sufficiently fine to capture the oscillatory behavior of the wave and hence, at high frequency and/or with a large number of obstacles, the size of the linear system becomes very large. Hence and due to their computational cost, direct solvers cannot be employed and the numerical solution is then handled by a Krylov subspace solver, such as GMRES [24], which can be improved via at least two major changes. First, the CPU cost of the matrix-vector product involved at each iteration can be reduced from $O(n^2)$ to $O(n \ln(n))$ thanks to Fast Multipole Method (FMM, see *e.g.* [12, 16, 17]). Secondly, the system can be preconditioned to enhance the convergence rate of the Krylov solver, which is greatly slowed down by the non-positiveness of the Helmholtz operator, especially at high frequency. Finding a robust and efficient preconditioner is non-trivial and is more complicated when the FMM is employed since the matrix of the system is not computed. In particular, only a few number

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of algebraic preconditioners can be applied, such as SPAI (SParse Approximate Inverse, see *e.g.* [18, 9]). Another possibility is to work with a well-conditioned integral equation and numerous analytic preconditioning techniques has already been proposed, as for example in [1, 2, 3, 15]. In multiple scattering context, a natural preconditioner is the one representing single scattering effects. For M obstacles and given the matrix of a discretized integral equation, this preconditioner is composed by the M blocks located on the diagonal of this matrix. As every block represent the scattering problem by one obstacle, this geometric preconditioner is called single scattering preconditioner.

This article focuses on the effects of this preconditioning on boundary integral equations. They are first studied on direct boundary integral equations, for which the unknown densities are exactly the Cauchy data. After being preconditioned by their single scattering preconditioner, it appears that the three classical direct integral equations become exactly the same! This surprising result is moreover independent of the geometry. Secondly, this preconditioner is applied to the indirect integral equation of Brakhage-Werner, where the unknown densities do not have a physical meaning anymore, and a similar result is obtained. More precisely, it turns out that the preconditioned Brakhage-Werner integral equation is similar¹ to the preconditioned direct integral equations. As a consequence, every integral equations will share the same spectral properties after being composed by their single scattering preconditioner, and the iterative solver will have the exact same convergence rate. These results are the central point of this article and, as far as the author knows, are new.

This paper begins by a general presentation of boundary integral formulations and a systematic way of building direct integral equations. This part is well-known but mandatory to prove the main results of this article. The first result, namely the equality between the preconditioned direct integral equation, is established in section 3. The case of the indirect integral equation of Brakhage-Werner is studied thereafter in section 4. Then in section 5, a numerical example using \mathbb{P}^1 finite elements discretization shows that the matrices of the discretized boundary integral equations share the same spectrum, and thus confirms the theoretical results. Finally, the paper ends with a short conclusion.

2 Classical direct integral equations

Here is presented a systematic method to obtain the usual boundary integral equation EFIE, MFIE and CFIE where the acronyms EFIE, MFIE and CFIE design here respectively *Electric Field Integral Equation*, *Magnetic Field Integral Equation* and *Combined Field Integral Equation*. This section is widely inspired by the lecture of Bendali and Fares [4] but is mandatory to introduce notations. More details can also be found for example in the the PhD thesis of the author [25] or of Darbas [14].

2.1 The scattering problem

Let the whole domain \mathbb{R}^d be filled with a homogeneous and nondissipative medium, where $d = 2, 3$ is the dimension of the problem. Let also Ω^- be a bounded open set of \mathbb{R}^d such that the propagation domain $\Omega^+ = \mathbb{R}^d \setminus \overline{\Omega^-}$ is connected. The boundary Γ of Ω^- is assumed to be smooth enough (say of class C^2) with a unit normal vector \mathbf{n} directed into Ω^+ . The illumination of the scatterer Ω^- by a time-harmonic incident wave u^{inc} gives rise to a scattered field u , solution of the following scattering problem (the time dependence is assumed to be of

¹In this paper, two bounded operators S and T acting on a Hilbert space X are called similar if there exists a invertibly bounded operator U on X such that $S = U^{-1}TU$.

the form $e^{-i\omega t}$ and the wavenumber k real and positive):

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega^+, \\ u = -u^{inc} & \text{on } \Gamma, \\ u \text{ outgoing,} \end{cases} \quad (1)$$

where $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and the outgoing condition stands for the Sommerfeld radiation condition:

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} \|\mathbf{x}\|^{(d-1)/2} \left(\nabla u \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - iku \right) = 0,$$

with $\|\mathbf{x}\| = (\sum_{j=1}^d x_j^2)^{1/2}$. This paper is restricted to a Dirichlet boundary condition (sound-soft obstacles) and an incident plane wave of direction β :

$$\forall \mathbf{x} \in \mathbb{R}^d, \quad u^{inc}(\mathbf{x}) = e^{ik\beta \cdot \mathbf{x}}.$$

However, the results presented in this article remain true with either a Neumann boundary condition and/or any incident wave smooth enough in a neighborhood of the obstacle Ω^- (e.g. a time harmonic wave emitted by point source, e.g. Green function). To obtain the direct integral equations, it is more convenient to work with the total field $u_T = u + u^{inc}$, solution of the following problem (u_T exists in Ω^+ since u^{inc} is plane and thus is a solution of the Helmholtz equation in \mathbb{R}^d)

$$\begin{cases} \Delta u_T + k^2 u_T = 0 & \text{in } \Omega^+, \\ u_T = 0 & \text{on } \Gamma, \\ (u_T - u^{inc}) \text{ outgoing.} \end{cases} \quad (2)$$

Recall that both problems (1) and (2) are uniquely solvable [13]:

Theorem 1. *The scattering problems (1) and (2) admit a unique solution.*

2.2 Integral operators

Let the volume single-layer integral operator \mathcal{L} be defined by (see e.g. [22, Theorem 6.12])

$$\begin{aligned} \mathcal{L} : H^{-1/2}(\Gamma) &\longrightarrow H_{loc}^1(\mathbb{R}^d) \\ \rho &\longmapsto \mathcal{L}\rho, \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad \mathcal{L}\rho(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) \, d\Gamma(\mathbf{y}), \end{aligned}$$

and the volume double-layer integral operator \mathcal{M} by

$$\begin{aligned} \mathcal{M} : H^{1/2}(\Gamma) &\longrightarrow H_{loc}^1(\mathbb{R}^d \setminus \Gamma) \\ \lambda &\longmapsto \mathcal{M}\lambda, \quad \forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma, \quad \mathcal{M}\lambda(\mathbf{x}) = - \int_{\Gamma} \partial_{\mathbf{n}_{\mathbf{y}}} G(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) \, d\Gamma(\mathbf{y}), \end{aligned}$$

where the spaces $H^{-1/2}(\Gamma)$, $H^1(\mathbb{R}^d \setminus \Gamma)$, $H_{loc}^1(\mathbb{R}^d \setminus \Gamma)$ are the usual Sobolev spaces and the Green function $G(\cdot, \cdot)$ is given by

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{y}, \quad G(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{i}{4} H_0^{(1)}(k\|\mathbf{x} - \mathbf{y}\|), & \text{if } d = 2, \\ \frac{e^{ik\|\mathbf{x} - \mathbf{y}\|}}{4\pi\|\mathbf{x} - \mathbf{y}\|}, & \text{if } d = 3. \end{cases} \quad (3)$$

Remark 2. All the integrals on Γ must be seen as a dual product between the Sobolev space $H^{1/2}(\Gamma)$ and its dual $H^{-1/2}(\Gamma)$. However, as soon as the data (u^{inc} and Γ) are smooth enough, then the scattered field u is also smooth and the dual product can be identified with the (non-hermitian) scalar product on $L^2(\Gamma)$:

$$\langle f, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \int_{\Gamma} f(\mathbf{x})g(\mathbf{x}) \, d\Gamma(\mathbf{x}).$$

This identification is considered throughout this paper.

The trace γ_0^{\pm} and the normal trace γ_1^{\pm} operators are now defined following and inspired by [11, Appendix A], where the plus or minus sign specifies whether the trace is taken from the inside of Ω^+ or Ω^- . First, the trace operators $\gamma_0^{\pm} : H^1(\Omega^{\pm}) \rightarrow H^{1/2}(\Gamma)$ are defined so that, if $v \in C^{\infty}(\overline{\Omega^{\pm}})$, then

$$\gamma_0^{\pm} v(\mathbf{x}) = \lim_{\mathbf{z} \in \Omega^{\pm} \rightarrow \mathbf{x}} v(\mathbf{z}),$$

for almost every $\mathbf{x} \in \Gamma$. By introducing the space $H^1(\Omega^{\pm}; \Delta) := \{v \in H^1(\Omega^{\pm}); \Delta v \in L^2(\Omega^{\pm})\}$ and the linear operators $\gamma_*^{\pm} : H^{1/2}(\Gamma) \rightarrow H^1(\Omega^{\pm})$ such that $\gamma_0^{\pm} \gamma_*^{\pm} \varphi = \varphi$, for all $\varphi \in H^{1/2}(\Gamma)$, the normal traces $\gamma_1^{\pm} : H^1(\Omega^{\pm}; \Delta) \rightarrow H^{-1/2}(\Gamma)$ can be defined [11, Equation (A.28)]:

$$\forall v \in H^1(\Omega^{\pm}; \Delta), \forall \varphi \in H^{1/2}(\Gamma),$$

$$\left(\gamma_1^{\pm} v, \varphi \right)_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} := \mp \left[\int_{\Omega^{\pm}} \Delta v(\mathbf{x}) \overline{w(\mathbf{x})} \, d\mathbf{x} + \int_{\Omega^{\pm}} \nabla v(\mathbf{x}) \cdot \nabla \overline{w(\mathbf{x})} \, d\mathbf{x} \right], \quad (4)$$

where $w := \gamma_*^{\pm} \varphi$ (and thus satisfies $\gamma_0^{\pm} w = \varphi$). As the quantities involved in scattering problem do not belong to $H^1(\Omega^+)$ but to $H_{loc}^1(\Omega^+)$, the exterior trace and normal trace operators are naturally extended as $\gamma_0^+ : H_{loc}^1(\Omega^+) \rightarrow H^{1/2}(\Gamma)$ and $\gamma_1^+ : H_{loc}^1(\Omega^+; \Delta) \rightarrow H^{-1/2}(\Gamma)$ by $\gamma_0^+(v) = \gamma_0^+(vv')$ and $\gamma_1^+(v) = \gamma_1^+(vv')$, where v' is an arbitrary compactly supported and indefinitely differentiable function on Ω^+ which is equal to 1 in a neighborhood of Γ , and where $H_{loc}^1(\Omega^+; \Delta) := \{v \in H_{loc}^1(\Omega^+); \Delta v \in L_{loc}^2(\Omega^+)\}$. Remark that, when the function v is sufficiently smooth, then its normal trace $\gamma_1^{\pm} v$, given by (4), belongs to $L^2(\Gamma)$ and can be written as $\gamma_1^{\pm} v(\mathbf{x}) = \lim_{\mathbf{z} \in \Omega^{\pm} \rightarrow \mathbf{x}} \nabla v(\mathbf{z}) \cdot \mathbf{n}(\mathbf{x})$, for almost every \mathbf{x} on Γ . Note also that, the single- and double-layer potentials, introduced previously, belong not only to $H_{loc}^1(\Omega^+) \cup H^1(\Omega^-)$ but also to $H_{loc}^1(\Omega^+; \Delta) \cup H^1(\Omega^-; \Delta)$ (see *e.g.* [11, §2.2]). Some well-known properties of the single- and double-layer potentials are summarized in the following propositions. Their proof can be found for example in [22, Theorems 7.5 and 9.6] for proposition 3 and in [22, Theorem 6.12] for proposition 4.

Proposition 3. For every densities $\rho \in H^{-1/2}(\Gamma)$ and $\lambda \in H^{1/2}(\Gamma)$, the single-layer potential $\mathcal{L}\rho$ and double-layer potential $\mathcal{M}\lambda$ are outgoing solutions of the Helmholtz equation in $\mathbb{R}^d \setminus \Gamma$. Moreover, the scattered field u , solution of (1), can be written as

$$\forall \mathbf{x} \in \Omega^+, \quad u(\mathbf{x}) = -\mathcal{L}(\partial_{\mathbf{n}} u|_{\Gamma})(\mathbf{x}) - \mathcal{M}(u|_{\Gamma})(\mathbf{x}).$$

Proposition 4. The trace and the normal trace of the operators \mathcal{L} and \mathcal{M} are given by the following relations

$$\begin{aligned} \gamma_0^{\pm} \mathcal{L}\rho &= L\rho, & \gamma_0^{\pm} \mathcal{M}\lambda &= \left(\mp \frac{1}{2} I + M \right) \lambda, \\ \gamma_1^{\pm} \mathcal{L}\rho &= \left(\mp \frac{1}{2} I + N \right) \rho, & \gamma_1^{\pm} \mathcal{M}\lambda &= D\lambda, \end{aligned} \quad (5)$$

where I is the identity operator and, for $\mathbf{x} \in \Gamma$, $\rho \in H^{-1/2}(\Gamma)$ and $\lambda \in H^{1/2}(\Gamma)$, the four boundary integral operators are defined by

$$\begin{aligned} L : H^{-1/2}(\Gamma) &\longrightarrow H^{1/2}(\Gamma), & L\rho(\mathbf{x}) &= \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\Gamma(\mathbf{y}), \\ M : H^{1/2}(\Gamma) &\longrightarrow H^{1/2}(\Gamma), & M\lambda(\mathbf{x}) &= - \int_{\Gamma} \partial_{\mathbf{n}_y} G(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) d\Gamma(\mathbf{y}), \\ N : H^{-1/2}(\Gamma) &\longrightarrow H^{-1/2}(\Gamma), & N\rho(\mathbf{x}) &= \int_{\Gamma} \partial_{\mathbf{n}_x} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\Gamma(\mathbf{y}) = -M^* \rho(\mathbf{x}), \\ D : H^{1/2}(\Gamma) &\longrightarrow H^{-1/2}(\Gamma), & D\lambda(\mathbf{x}) &= -\partial_{\mathbf{n}_x} \int_{\Gamma} \partial_{\mathbf{n}_y} G(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) d\Gamma(\mathbf{y}). \end{aligned} \tag{6}$$

In this paper, the boundary integral operators are written with a roman letter (*e.g.* L) whereas the volume integral operators are written with a calligraphic letter (*e.g.* \mathcal{L}). According to [23, Theorems 3.4.1 and 3.4.2], the boundary integral operators L and D are invertible, providing k is not an irregular frequency.

Theorem 5. *Let $F_D(\Omega^-)$ (resp. $F_N(\Omega^-)$) be the countable set of positive wavenumbers k accumulating at infinity such that the interior homogeneous Dirichlet (resp. Neumann) problem*

$$\begin{cases} -\Delta v = k^2 v & \text{in } \Omega^-, \\ v = 0 \text{ (resp. } \partial_{\mathbf{n}} v = 0) & \text{on } \Gamma, \end{cases} \tag{7}$$

admits non-trivial solutions. Then, the operator L (resp. D) realizes an isomorphism from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ (resp. from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$) if and only if $k \notin F_D(\Omega^-)$ (resp. $k \notin F_N(\Omega^-)$).

These irregular frequencies k of $F_D(\Omega^-)$ (resp. of $F_N(\Omega^-)$) are exactly the square roots of the eigenvalues of the Laplacian operator $(-\Delta)$ for the homogeneous interior Dirichlet (resp. Neumann) problem. In the multiple scattering case, that is when $\Omega^- = \bigcup_{p=1}^M \Omega_p^-$ is multiply connected, the following equalities clearly hold true

$$F_D(\Omega^-) = \bigcup_{p=1}^M F_D(\Omega_p^-) \quad \text{and} \quad F_N(\Omega^-) = \bigcup_{p=1}^M F_N(\Omega_p^-). \tag{8}$$

Throughout the paper, $F_{DN}(\Omega^-)$ denotes the set of all irregular frequencies:

$$F_{DN}(\Omega^-) = F_D(\Omega^-) \bigcup F_N(\Omega^-). \tag{9}$$

2.3 Direct integral equations

This section details the way of deriving direct integral equations, described in [4]. This approach is nonstandard but has advantages that appear later in the paper at Section 3.

The principle is to write the total field u_T as a combination of a single- and a double-layer potentials:

$$u_T(\mathbf{x}) = \mathcal{L}\rho(\mathbf{x}) + \mathcal{M}\lambda(\mathbf{x}) + u^{inc}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega^+, \tag{10}$$

where (λ, ρ) are now the two unknown of the problem. Thanks to proposition 3, such an expression ensures that both u_T is solution of the Helmholtz equation in Ω^+ and $(u_T - u^{inc})$ is outgoing. Following [4], an integral equation is said to be direct when the densities (λ, ρ) have a physical meaning. Indeed, for these integral equations, they are exactly the Cauchy

data $(-u_T|_\Gamma, -\partial_{\mathbf{n}}u_T|_\Gamma)$. However, this is not a choice but a consequence of the construction of the integral equation. In electromagnetic scattering, direct and indirect integral equations are more often referred to as respectively *field* and *source* integral equations (see *e.g.* Harington and Mautz [19, 21] or PhD thesis of Sophie Borel [6]).

For now on, the problem, composed by the two unknown (λ, ρ) , has only one equation given by the Dirichlet boundary condition on Γ . To obtain a second equation, a fictitious interior wave u_T^- , living in Ω^- , is introduced and defined by

$$u_T^-(\mathbf{x}) = \mathcal{L}\rho(\mathbf{x}) + \mathcal{M}\lambda(\mathbf{x}) + u^{inc}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega^-. \quad (11)$$

Remark that, on the one hand u_T^- is a solution of the Helmholtz equation in Ω^- and on the other hand, due to the trace relations (5), the couple of unknown (λ, ρ) satisfies the well-known *jump-relation*

$$\begin{cases} \lambda = u_T^-|_\Gamma - u_T|_\Gamma, \\ \rho = \partial_{\mathbf{n}}u_T^-|_\Gamma - \partial_{\mathbf{n}}u_T|_\Gamma. \end{cases} \quad (12)$$

As the wave u_T^- is fictitious, it does not act on the solution u_T of the scattering problem. As a consequence, the boundary condition on Γ imposed to u_T^- has no influence on u_T . Let this constraint be represented by an operator A such that u_T^- is the solution of the following interior problem

$$\begin{cases} \Delta u_T^- + k^2 u_T^- = 0 & \text{in } \Omega^-, \\ Au_T^- = 0 & \text{on } \Gamma. \end{cases} \quad (13)$$

To build a direct integral equation, the operator A is chosen such that the field u_T^- vanishes in Ω^- . Supposing that such an operator exists, then, on the boundary Γ , the following equalities will hold true

$$\begin{cases} u_T^-|_\Gamma = 0, \\ \partial_{\mathbf{n}}u_T^-|_\Gamma = 0. \end{cases}$$

Consequently and thanks to the Dirichlet boundary condition $u_T|_\Gamma = 0$, the jump relations (12) will read as

$$\begin{cases} \lambda = 0, \\ \rho = -\partial_{\mathbf{n}}u_T|_\Gamma, \end{cases}$$

Therefore, both the fictitious field u_T^- and the total field u_T will be composed by a single-layer potential only

$$\begin{cases} u_T(\mathbf{x}) = \mathcal{L}\rho(\mathbf{x}) + u^{inc}(\mathbf{x}), & \forall \mathbf{x} \in \Omega^+, \\ u_T^-(\mathbf{x}) = \mathcal{L}\rho(\mathbf{x}) + u^{inc}(\mathbf{x}), & \forall \mathbf{x} \in \Omega^-. \end{cases}$$

The unknown ρ is finally obtained through the resolution of the (direct) integral equation $Au_T^- = 0$, which can be written as

$$A\mathcal{L}\rho = -Au^{inc}. \quad (14)$$

Both the expression and the nature of the integral equation (14) depend on the boundary condition imposed to u_T^- , represented here by the operator A . The next subsections describe the three usual direct integral equations that are studied in this paper. The proofs are not provided and can be found for example in [4] or [25].

2.3.1 EFIE (*Electric Field Integral Equation*)

For this integral equation, the operator A is the interior trace operator γ_0^- on Γ . Thanks to the continuity on Γ of the single-layer integral operator \mathcal{L} (see equation (5)), the boundary integral equation (14) becomes

$$L\rho = -u^{inc}|_{\Gamma}. \quad (15)$$

Due to theorem 5, this first kind integral equation, named *Electric Field Integral Equation* (EFIE), is well-posed and equivalent to the scattering problem (2) except for Dirichlet irregular frequencies.

Proposition 6. *If $k \notin F_D(\Omega^-)$ then the single-layer potential $\mathcal{L}\rho + u^{inc}$ is solution of the scattering problem (2) if and only if ρ is the solution of the EFIE (15).*

Remark 7. *When $k \in F_D(\Omega^-)$, the integral operator L is no more bijective but is still one-to-one. It can be shown that the kernel of the operator L is a subset of the kernel of the operator \mathcal{L} . Consequently, for every solution $\tilde{\rho}$ of the EFIE, the associated single-layer potential $\mathcal{L}\tilde{\rho} + u^{inc}$ is still the solution of the scattering problem (2).*

2.3.2 MFIE (*Magnetic Field Integral Equation*)

Another possibility is to chose $A = \gamma_1^-$, the interior normal trace. Using traces formulæ (5), the integral equation (14) becomes

$$\left(\frac{1}{2}I + N\right)\rho = -\partial_{\mathbf{n}}u^{inc}|_{\Gamma}. \quad (16)$$

This Fredholm second kind integral equation, named *Magnetic Field Integral Equation* (MFIE), is well-posed and equivalent to the scattering problem (2) as far as k is not an irregular Neumann frequency.

Proposition 8. *If $k \notin F_N(\Omega^-)$, then the quantity $\mathcal{L}\rho + u^{inc}$ is the solution of the scattering problem (2) if and only if ρ is the solution of the MFIE (16).*

Remark 9. *For every irregular frequency k of $F_N(\Omega^-)$, the operator $\left(\frac{1}{2}I + N\right)$ is no more one-to-one. In that case and contrary to the EFIE, the single-layer potential $\mathcal{L}\tilde{\rho} + u^{inc}$ based on a solution $\tilde{\rho}$ of the MFIE is not guaranteed to be the solution of the scattering problem (2).*

2.3.3 CFIE (*Combined Field Integral Equation*)

To avoid the irregular frequencies problem, Burton and Miller [8] considered a linear combination of the EFIE and the MFIE by imposing a Fourier-Robin boundary condition to u_T^- on Γ :

$$A = (1 - \alpha)\gamma_1^- + \alpha\eta\gamma_0^-,$$

with

$$0 < \alpha < 1 \quad \text{and} \quad \Im(\eta) \neq 0, \quad (17)$$

where $\Im(\eta)$ is the imaginary part of the complex number η . Hence, the boundary integral equation (14) reads as

$$\left[(1 - \alpha)\left(\frac{1}{2}I + N\right) + \alpha\eta L\right]\rho = -\left[(1 - \alpha)\partial_{\mathbf{n}}u^{inc}|_{\Gamma} + \alpha\eta u^{inc}|_{\Gamma}\right]. \quad (18)$$

This *Combined Field Integral Equation* (CFIE, denomination of Harrington and Mautz [19] in electromagnetism) or Burton-Miller integral equation [8] is well-posed for every frequency k .

Proposition 10. *For any $k > 0$ and for any couple α and η satisfying condition (17), the single-layer potential $\mathcal{L}\rho + u^{inc}$ is the solution of the scattering problem if and only if ρ is the solution of the CFIE (2).*

3 Single scattering preconditioned direct integral equations

3.1 Integral equation associated to A

Previous sections show that depending on the choice of the operator A , the boundary integral equation (14) will be the EFIE (trace), the MFIE (normal trace) or the CFIE (linear combination). Therefore, in order to merge the notations, only the boundary integral equation (14) will be considered, which can be rewritten as

$$L_A \rho = -A u^{inc}, \quad \text{with } L_A \rho = A \mathcal{L} \rho. \quad (19)$$

In this paper, this integral equation is called *integral equation associated to A* or, in short, *integral equation A* . It represents one of the three direct integral equation EFIE, MFIE or CFIE, depending on the choice of A . As an example, for the EFIE, the operator A is the interior trace γ_0^- and $L_A = \gamma_0^- \mathcal{L} = L$ is the boundary single-layer integral operator. According to Properties 6, 8 and 10, the integral equation associated to A is uniquely solvable if $k \notin F_{DN}(\Omega^-)$.

Proposition 11. *Let us assume that $k \notin F_{DN}(\Omega^-)$ and that A is whether the interior trace γ_0^- , the interior normal trace γ_1^- or a linear combination $(1 - \alpha)\gamma_1^- + \alpha\eta\gamma_0^-$ such that α and η satisfy relation (17). The quantity $\mathcal{L}\rho + u^{inc}$ is then the unique solution of the scattering problem (2) if and only if ρ is the unique solution of the integral equation A (19).*

In what follows, irregular frequencies will be avoided by always assuming that $k \notin F_{DN}(\Omega^-)$.

3.2 Multiple scattering case

The domain Ω^- is now supposed to be a collection of M disjoint bounded open sets Ω_p^- of \mathbb{R}^d , $p = 1, \dots, M$, such that every domain $(\mathbb{R}^d \setminus \overline{\Omega_p^-})$ is connected, as this is the case for the propagation domain $\Omega^+ = \mathbb{R}^d \setminus \overline{\Omega^-}$. In this paper, single scattering designates scattering in a medium containing only one scatterer whereas multiple scattering is used for a medium containing more than one obstacle. This article being focused on the multiple scattering case, M will be assumed to satisfy $M \geq 2$.

As Ω^- is composed of M disjoint obstacles Ω_p^- , $p = 1, \dots, M$, the single-layer volume integral operator \mathcal{L} can be written as the sum of M operators \mathcal{L}_q , $q = 1, \dots, M$, defined by

$$\begin{aligned} \mathcal{L}_q : H^{-1/2}(\Gamma_q) &\longrightarrow H_{loc}^1(\mathbb{R}^d) \\ \rho_q &\longmapsto \mathcal{L}_q \rho_q, \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad \mathcal{L}_q \rho_q(\mathbf{x}) = \int_{\Gamma_q} G(\mathbf{x}, \mathbf{y}) \rho_q(\mathbf{y}) \, d\mathbf{y}. \end{aligned} \quad (20)$$

Therefore the single-layer potential can be decomposed as follows

$$\forall \rho \in H^{-1/2}(\Gamma), \quad \mathcal{L} \rho = \sum_{q=1}^M \mathcal{L}_q \rho_q, \quad \text{with } \rho_q = \rho|_{\Gamma_q}. \quad (21)$$

Now, for every $p = 1, \dots, M$, let A_p be the restriction of the operator A to Γ_p :

$$\forall g \in H^1(\Omega^-), \quad A_p g = (Ag)|_{\Gamma_p}, \quad (22)$$

With these notations, the integral equation A (19) satisfied by ρ can be written equivalently as a system of M integral equations

$$\forall p = 1, \dots, M, \quad A_p \mathcal{L} \rho = -A_p u^{inc},$$

or, using the decomposition (21), as

$$\forall p = 1, \dots, M, \quad \sum_{q=1}^M A_p \mathcal{L}_q \rho_q = -A_p u^{inc}. \quad (23)$$

Finally, by introducing the operators $L_A^{p,q}$, for $p, q = 1, \dots, M$, defined by

$$\forall \rho_q \in H^{-1/2}(\Gamma_q), \quad L_A^{p,q} \rho_q = A_p(\mathcal{L}_q \rho_q), \quad (24)$$

then the M integral equations (23) can be written in the following matrix form

$$\begin{bmatrix} L_A^{1,1} & L_A^{1,2} & \dots & L_A^{1,M} \\ L_A^{2,1} & L_A^{2,2} & \dots & L_A^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ L_A^{M,1} & L_A^{M,2} & \dots & L_A^{M,M} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_M \end{bmatrix} = - \begin{bmatrix} A_1 u^{inc} \\ A_2 u^{inc} \\ \vdots \\ A_M u^{inc} \end{bmatrix}.$$

Recall that this equation is just a matrix form of the integral equation associated to A (19). For now on, the operator L_A will be identified to its associated matrix $(L_A^{p,q})_{1 \leq p, q \leq M}$.

3.3 Single scattering operator and preconditioned integral equation A

Let the *single scattering operator* \hat{L}_A , corresponding to the diagonal part of the operator L_A , be defined by:

$$\hat{L}_A = \begin{bmatrix} L_A^{1,1} & 0 & \dots & 0 \\ 0 & L_A^{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_A^{M,M} \end{bmatrix}. \quad (25)$$

Indeed, each component $L_A^{p,p}$ of \hat{L}_A represents the self-interaction of the scatterer Ω_p^- . More precisely, if the medium contains only one obstacle Ω_p^- , with $p \in \{1, \dots, M\}$, then this equality would hold true $L_A = L_A^{p,p}$.

As the wavenumber k is assumed not to be an irregular frequency of $F_{DN}(\Omega^-)$, then k does also not belong to $F_{DN}(\Omega_p^-)$, for all $p = 1, \dots, M$, thanks to relations (8) and (9). Consequently and due to proposition 11, for $p = 1, \dots, M$, the operator $L_A^{p,p}$ associated to the single scattering is invertible. Thus, the single scattering operator \hat{L}_A is also invertible with inverse operator

$$\hat{L}_A^{-1} = \begin{bmatrix} (L_A^{1,1})^{-1} & 0 & \dots & 0 \\ 0 & (L_A^{2,2})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (L_A^{M,M})^{-1} \end{bmatrix}.$$

The integral equation associated to A is now preconditioned by \hat{L}_A which gives rise to the *preconditioned integral equation associated to A* (or in short *the preconditioned integral equation A*):

$$\hat{L}_A^{-1} L_A \rho = -\hat{L}_A^{-1} A u^{inc}, \quad (26)$$

where the operator $\widehat{L}_A^{-1}L_A$ has the following matrix form

$$\widehat{L}_A^{-1}L_A = \begin{bmatrix} I & (L_A^{1,1})^{-1}L_A^{1,2} & \dots & (L_A^{1,1})^{-1}L_A^{1,M} \\ (L_A^{2,2})^{-1}L_A^{2,1} & I & \dots & (L_A^{2,2})^{-1}L_A^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ (L_A^{M,M})^{-1}L_A^{M,1} & (L_A^{M,M})^{-1}L_A^{M,2} & \dots & I \end{bmatrix}. \quad (27)$$

Note that this preconditioning accelerates the convergence rate of an iterative solver, like the GMRES, as illustrated by the numerical example given in Section 5.

3.4 Equality of the preconditioned direct integral equations

This section contains the main result of the paper, that is the equality between the three direct preconditioned integral equations. In other words, here is shown that the preconditioned integral equations associated to A is independent of the choice of the operator A . As far as the author knows, this surprising result is new.

To prove this, a second “general” integral equation, called *integral equation associated to B* , with $B \neq A$, is introduced and preconditioned in the same way as the integral equation associated to A :

$$\widehat{L}_B^{-1}L_B\rho = -\widehat{L}_B^{-1}Bu^{inc}, \quad (28)$$

where $L_B = B\mathcal{L}$ and the operator B is whether the interior trace γ_0^- , the interior normal trace γ_1^- or a linear combination $(1-\alpha)\gamma_1^- + \eta\alpha\gamma_0^-$ with α and η satisfying (17). The operators \widehat{L}_B is defined in the same way as the operator \widehat{L}_A (see relation (25)), with a formal change of A by B . Moreover and similarly to relations (22) and (24), the operators B_p and $L_B^{p,q}$, $p, q = 1, \dots, M$, are introduced and defined by

$$\forall g \in H^1(\Omega^-), \quad B_p g = (Bg)|_{\Gamma_p} \quad \text{and} \quad \forall \rho_q \in H^{-1/2}(\Gamma_q), \quad L_B^{p,q} \rho_q = B_p \mathcal{L}_q \rho_q.$$

Remark that, due to proposition 11, for $k \notin F_{DN}(\Omega^-)$, the quantity $\mathcal{L}\rho + u^{inc}$ is the solution of the scattering problem (2) if and only if ρ is the solution of the integral equation associated to A or to B . In other words, provided $k \notin F_{DN}(\Omega^-)$, the two integral equations A and B are equivalent and have the same solution.

The aim of this section is to prove that the preconditioned integral equation associated to A (26) is exactly the same as the preconditioned integral equation associated to B (28). Since the unknown ρ is the same in both equations, it is sufficient to prove that the operators $\widehat{L}_A^{-1}L_A$ and $\widehat{L}_B^{-1}L_B$ are identical. More precisely and thanks to their matrix form (27), it is sufficient to show that the following equality holds true

$$\forall p, q = 1, \dots, M, \quad (L_A^{p,p})^{-1}L_A^{p,q} = (L_B^{p,p})^{-1}L_B^{p,q}. \quad (29)$$

Furthermore, for $p = q$, the above equality is obvious since

$$(L_A^{p,p})^{-1}L_A^{p,p} = (L_B^{p,p})^{-1}L_B^{p,p} = I_p,$$

where I_p is the identity operator on $H^{-1/2}(\Gamma_p)$. Therefore, equality (29) needs to be shown only for $p \neq q$. Let the following result be first established.

Lemma 12. *Assuming that $k \notin F_{DN}(\Omega^-)$ then the following equality holds true*

$$\forall p = 1, \dots, M, \quad (L_A^{p,p})^{-1}A_p = (L_B^{p,p})^{-1}B_p.$$

Proof. Let $p = 1, \dots, M$ and g be an element of $H_{loc}^1(\mathbb{R}^d)$. The single scattering problem associated to the obstacle Ω_p^- is to find the scattered field v solution of

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega_p^-}, \\ v = -g & \text{on } \Gamma_p, \\ v \text{ outgoing.} \end{cases} \quad (30)$$

The unique solution of this problem is the single-layer potential $\mathcal{L}_p \rho_p$, where $\rho_p \in H^{-1/2}(\Gamma_p)$ is indifferently obtained by solving the integral equation associated to A

$$L_A^{p,p} \rho_p = -A_p g,$$

or the one associated to B

$$L_B^{p,p} \rho_p = -B_p g.$$

These two equations being well-posed ($k \notin F_{DN}(\Omega^-)$), we have

$$(L_A^{p,p})^{-1} A_p g = (L_B^{p,p})^{-1} B_p g.$$

The proof is ended by virtue of the arbitrariness of g . \square

The main result of this section can now be established.

Theorem 13. *If $k \notin F_{DN}(\Omega^-)$, then the operators $\hat{L}_A^{-1} L_A$ and $\hat{L}_B^{-1} L_B$ are equal. In other words, the operator $\hat{L}_A^{-1} L_A$ does not depend on the choice of the operator A .*

Proof. It is sufficient to show that, for $p, q = 1, \dots, M$, with $p \neq q$,

$$(L_A^{p,p})^{-1} L_A^{p,q} = (L_B^{p,p})^{-1} L_B^{p,q}.$$

Recalling that the operator $L_A^{p,q}$ is given by $L_A^{p,q} = A_p \mathcal{L}_q$ and using lemma 12, it appears that

$$(L_A^{p,p})^{-1} L_A^{p,q} = (L_A^{p,p})^{-1} A_p \mathcal{L}_q = (L_B^{p,p})^{-1} B_p \mathcal{L}_q.$$

Therefore, applying the definition $B_p \mathcal{L}_q = L_B^{p,q}$ gives rise to the sought equality

$$(L_A^{p,p})^{-1} L_A^{p,q} = (L_B^{p,p})^{-1} L_B^{p,q}, \quad \forall p, q = 1, \dots, M,$$

and the proof is ended. \square

As a conclusion, let us point out that the preconditioned integral equations A and B have the same operator ($\hat{L}_A^{-1} L_A = \hat{L}_B^{-1} L_B$) and the same solution ρ . Consequently, their right hand sides are also equal:

$$-\hat{L}_A^{-1} u^{inc} = -\hat{L}_B^{-1} u^{inc}.$$

Thus, the preconditioned integral equation A

$$\hat{L}_A^{-1} L_A \rho = -\hat{L}_A^{-1} u^{inc},$$

and the preconditioned integral equation B

$$\hat{L}_B^{-1} L_B \rho = -\hat{L}_B^{-1} u^{inc},$$

are exactly the same. As a consequence, preconditioning any direct integral equation EFIE, MFIE or CFIE with its single scattering operator will lead to exactly the same equation. Obviously, the frequency k must not be an irregular one, because in that case the single scattering operator \hat{L}_A could be no more invertible (especially for the EFIE and the MFIE). Finally, it should be pointed out that the CFIE is well-posed for every wavenumber $k > 0$ and thus, the preconditioned CFIE is also well-posed for all $k > 0$.

Remark 14. *Theorem 13 is written for the three direct integral equations EFIE, MFIE and CFIE, but can be easily extended to any other direct boundary integral equation, for which the scattered field is also given by $u = \mathcal{L}\rho$, with $\rho = -\partial_{\mathbf{n}}u_T|_{\Gamma}$. In fact, theorem 13 can be proved for any couple of boundary integral equations, provided that they share the same expression of the scattered field and that they are based on the exact same unknown density (e.g. single-layer potential of density $\rho = -\partial_{\mathbf{n}}u_T|_{\Gamma}$ for the direct integral equations). In other words, with these assumptions and after being preconditioned by their single scattering operator, these set of boundary integral equations will become identical to each other.*

4 Brakhage-Werner indirect integral equation

The above results are now extended to the Brakhage-Werner indirect integral equation [7]. In fact, after being preconditioned by its single scattering operator, the Brakhage-Werner integral equation does not lead to the same equation as the one obtained with the direct integral equations. However they are similar. This section begins by recalling how the Brakhage-Werner integral equation can be obtained and after that the result is established by comparison with the EFIE.

4.1 Brakhage-Werner indirect integral equation

This paragraph begins with some notations. The total field u_T is here sought as a linear combination of a single- and a double-layer of density $\psi \in H^{1/2}(\Gamma)$:

$$u_T = u^{inc} + \mathcal{L}_{BW}\psi,$$

where the operator \mathcal{L}_{BW} of parameter η_{BW} is given by

$$\forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma, \quad \mathcal{L}_{BW}\psi(\mathbf{x}) = (-\eta_{BW}\mathcal{L} - \mathcal{M})\psi(\mathbf{x}) = \int_{\Gamma} \left(\partial_{\mathbf{n}_{\mathbf{y}}} G(\mathbf{x}, \mathbf{y}) - \eta_{BW} G(\mathbf{x}, \mathbf{y}) \right) \psi(\mathbf{y}) \, d\Gamma(\mathbf{y}), \quad (31)$$

with $\Im(\eta) \neq 0$. The integral equation is obtained by applying the exterior trace γ_0^+ on Γ to u_T . Indeed, the Dirichlet boundary condition $\gamma_0^+ u_T = 0$ and the traces relation (5) directly give the Brakhage-Werner integral equation solved by ψ

$$L_{BW}\psi = -u^{inc}|_{\Gamma}, \quad (32)$$

with

$$L_{BW} = \left(-\eta L - M + \frac{1}{2}I \right).$$

This second kind integral equation does not suffer from irregular frequency [7].

Proposition 15. *For all $k > 0$, the quantity $\mathcal{L}_{BW}\psi + u^{inc}$ is the solution of the scattered field (2) if and only if ψ is the solution of the Brakhage-Werner integral equation (32).*

Remark 16. *Other generalizations of these equations, when η_{BW} is an operator, are available for example in [1, 2, 3]*

Remark 17. *A numerical study concerning the optimal choice of parameter η_{BW} , appearing in relation (31), is proposed in [20] in the case of a single spherical or circular obstacle of radius R . For a Dirichlet boundary condition, the choice $\eta_{BW} = i/2 \max(1/R, k)$ leads to a reasonable condition number of the matrix of the linear system associated to the Brakhage-Werner integral equation, for sufficiently high frequency. Recent works have been done on how to choose this*

parameter for much more general domains, see for example [10, §6] and [11, §5.1] for the case of large k and [5, §2.6 and §2.7] for the case of small frequency k . Note also that, according to [11, Remark 2.24], these results apply to both L_{BW} and the CFIE operator, since when $\alpha = 1/2$, these operators are adjoints (up to a factor of $1/2$) in the real L^2 inner product.

As in the previous section, M volume integral operators \mathcal{L}_{BW}^q , for $q = 1, \dots, M$, are introduced and defined for every density $\psi_q \in H^{1/2}(\Gamma_q)$ by

$$\forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma_q, \quad \mathcal{L}_{BW}^q \psi_q(\mathbf{x}) = \int_{\Gamma_q} \left(\partial_{\mathbf{n}_y} G(\mathbf{x}, \mathbf{y}) - \eta G(\mathbf{x}, \mathbf{y}) \right) \psi_q(\mathbf{y}) \, d\Gamma(\mathbf{y}).$$

Thus, the potential $\mathcal{L}_{BW}\psi$ reads as

$$\forall \psi \in H^{1/2}(\Gamma), \quad \mathcal{L}_{BW}\psi = \sum_{q=1}^M \mathcal{L}_{BW}^q \psi_q, \quad \text{with } \psi_q = \psi|_{\Gamma_q}.$$

Finally, for $p, q = 1, \dots, M$, the operator $L_{BW}^{p,q}$ is defined by

$$\forall \psi_q \in H^{1/2}(\Gamma_q), \quad L_{BW}^{p,q} \psi_q = (\mathcal{L}_{BW}^q \psi_q)|_{\Gamma_p}. \quad (33)$$

The Brakhage-Werner integral equation (32) can now be written in the following matrix form

$$\begin{bmatrix} L_{BW}^{1,1} & L_{BW}^{1,2} & \dots & L_{BW}^{1,M} \\ L_{BW}^{2,1} & L_{BW}^{2,2} & \dots & L_{BW}^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ L_{BW}^{M,1} & L_{BW}^{M,2} & \dots & L_{BW}^{M,M} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_M \end{bmatrix} = - \begin{bmatrix} u^{inc}|_{\Gamma_1} \\ u^{inc}|_{\Gamma_2} \\ \vdots \\ u^{inc}|_{\Gamma_M} \end{bmatrix}, \quad (34)$$

and the single scattering operator \hat{L}_{BW} of the Brakhage-Werner integral equation reads as

$$\hat{L}_{BW} = \begin{bmatrix} L_{BW}^{1,1} & 0 & \dots & 0 \\ 0 & L_{BW}^{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_{BW}^{M,M} \end{bmatrix}.$$

Because of the well-posedness of the Brakhage-Werner integral equation for every frequency k , each operator $L_{BW}^{p,p}$ is invertible. Thus, the operator \hat{L}_{BW} is invertible with inverse

$$\hat{L}_{BW}^{-1} = \begin{bmatrix} (L_{BW}^{1,1})^{-1} & 0 & \dots & 0 \\ 0 & (L_{BW}^{2,2})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (L_{BW}^{M,M})^{-1} \end{bmatrix}.$$

The Brakhage-Werner integral equation (32) can now be composed on its left by the operator \hat{L}_{BW}^{-1} to obtain the *preconditioned Brakhage-Werner integral equation*

$$\hat{L}_{BW}^{-1} L_{BW} \psi = -\hat{L}_{BW}^{-1} (u^{inc}|_{\Gamma}). \quad (35)$$

4.2 EFIE

Let the EFIE, given by equation (15), be considered again. For this integral equation, the total field u_T is written as a single-layer potential with density $\rho \in H^{-1/2}(\Gamma)$:

$$u_T = u^{inc} + \mathcal{L}\rho.$$

The EFIE can be obtained through (at least) two possibilities. The first one, considered in the previous section, consists in introducing a fictitious interior field $u_T^- = \mathcal{L}\rho + u^{inc}$ in Ω^- and applying to it a homogeneous Dirichlet boundary condition on Γ . Another possibility is to apply directly the Dirichlet boundary condition $u_T|_\Gamma = 0$ to the quantity $u^{inc} + \mathcal{L}\rho$. Thanks to the continuity of the single-layer potential through Γ (Proposition 4), this gives directly the electric field integral equation:

$$L\rho = -u^{inc}|_\Gamma. \quad (36)$$

As a consequence, both the EFIE and the Brakhage-Werner integral equation are obtained through a direct application of the exterior trace γ_0^+ to the total field u_T . This common point is the key to prove that, after being preconditioned by their single scattering operator, the EFIE and the Brakhage-Werner integral equation are similar. In what follows, the wavenumber k is assumed to satisfy $k \notin F_D(\Omega^-)$, which implies that the operator L is invertible. The volume single-layer integral operator \mathcal{L} is now decomposed into M operators $(\mathcal{L}_q)_{1 \leq q \leq M}$, as previously (see relation (20)), which leads to introduce the operators $(L^{p,q})_{1 \leq p,q \leq M}$ defined by

$$\forall \rho_q \in H^{-1/2}(\Gamma_q), \quad L^{p,q}\rho_q = (\mathcal{L}_q\rho_q)|_{\Gamma_p}. \quad (37)$$

With these notations, the EFIE has the following matrix form

$$\begin{bmatrix} L^{1,1} & L^{1,2} & \dots & L^{1,M} \\ L^{2,1} & L^{2,2} & \dots & L^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ L^{M,1} & L^{M,2} & \dots & L^{M,M} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_M \end{bmatrix} = - \begin{bmatrix} u^{inc}|_{\Gamma_1} \\ u^{inc}|_{\Gamma_2} \\ \vdots \\ u^{inc}|_{\Gamma_M} \end{bmatrix}, \quad (38)$$

and its single scattering operator \hat{L} is given by

$$\hat{L} = \begin{bmatrix} L^{1,1} & 0 & \dots & 0 \\ 0 & L^{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L^{M,M} \end{bmatrix}.$$

According to theorem 5, the operator \hat{L} is invertible. Therefore, the EFIE can be left composed by \hat{L}^{-1} which gives rise to the preconditioned EFIE

$$\hat{L}^{-1}L\rho = -\hat{L}^{-1}u^{inc}|_\Gamma, \quad (39)$$

with

$$\hat{L}^{-1} = \begin{bmatrix} (L^{1,1})^{-1} & 0 & \dots & 0 \\ 0 & (L^{2,2})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (L^{M,M})^{-1} \end{bmatrix}.$$

4.3 Relation between preconditioned EFIE and preconditioned Brakhage-Werner integral equation

In this section is proved that the operators $\hat{L}^{-1}L$ and $\hat{L}_{BW}^{-1}L_{BW}$ are similar. To achieve this, the following lemma must first be established.

Lemma 18. *When $k \notin F_D(\Omega^-)$, then for $p = 1, \dots, M$, the below equality holds true*

$$\mathcal{L}_p(L^{p,p})^{-1} = \mathcal{L}_{BW}^p(L_{BW}^{p,p})^{-1}.$$

Proof. For $p = 1, \dots, M$, consider the scatterer Ω_p^- , an element f_p of $H^{1/2}(\Gamma_p)$ and the following single scattering problem of unknown field v

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega_p^-}, \\ v = -f_p & \text{on } \Gamma_p, \\ v \text{ outgoing.} \end{cases} \quad (40)$$

Let ω and ω_{BW} be two potentials defined by

$$\begin{cases} \omega = \mathcal{L}_p[(L^{p,p})^{-1}f_p], \\ \omega_{BW} = \mathcal{L}_{BW}^p[(L_{BW}^{p,p})^{-1}f_p]. \end{cases}$$

As a linear combination of single- and double-layer potential, these two functions are radiating solution of the Helmholtz equation in $(\mathbb{R}^d \setminus \overline{\Omega_p^-})$ (see proposition 3). Moreover, applying the exterior trace on Γ_p to ω and ω_{BW} leads to

$$\begin{cases} \omega|_{\Gamma_p} = -(\mathcal{L}_p(L^{p,p})^{-1}f_p)|_{\Gamma_p} = -L^{p,p}(L^{p,p})^{-1}f_p = -f_p, \\ \omega_{BW}|_{\Gamma_p} = -(\mathcal{L}_{BW}^p(L_{BW}^{p,p})^{-1}f_p)|_{\Gamma_p} = -L_{BW}^{p,p}(L_{BW}^{p,p})^{-1}f_p = -f_p. \end{cases}$$

Thus, the waves ω and ω_{BW} are both solutions of the single scattering problem (40). By unicity (see theorem 1), it appears that

$$\forall f_p \in H^{1/2}(\Gamma_p), \quad \mathcal{L}_p(L^{p,p})^{-1}f_p = \mathcal{L}_{BW}^p(L_{BW}^{p,p})^{-1}f_p.$$

□

A second lemma can now be obtained.

Lemma 19. *For $k \notin F_D(\Omega^-)$, the following equality holds true*

$$L\hat{L}^{-1} = L_{BW}\hat{L}_{BW}^{-1}.$$

Proof. Thanks to the matrix form of the operators (see relations (38) and (34)), it is more convenient to prove that

$$\forall p, q = 1, \dots, M, \quad L^{q,p}(L^{p,p})^{-1} = L_{BW}^{q,p}(L_{BW}^{p,p})^{-1}. \quad (41)$$

For two indices $1 \leq p, q \leq M$, lemma 18 implies that

$$\mathcal{L}_p(L^{p,p})^{-1} = \mathcal{L}_{BW}^p(L_{BW}^{p,p})^{-1}.$$

Applying the exterior trace on Γ_q to this equality and using relations (37) and (33) lead to

$$L^{q,p}(L^{p,p})^{-1} = L_{BW}^{q,p}(L_{BW}^{p,p})^{-1},$$

which ends the proof. □

For now on, ρ denotes the (unique) solution of the EFIE (36) and ψ the one of the Brakhage-Werner integral equation (32). In other words, this means that

$$L\rho = -u^{inc}|_{\Gamma} \quad \text{and} \quad L_{BW}\psi = -u^{inc}|_{\Gamma}.$$

Consequently, the densities ρ and ψ are linked through the following relation

$$\rho = L^{-1}L_{BW}\psi.$$

The next proposition shows that the invertible operator $L^{-1}L_{BW}$ is the transformation operator between the preconditioned EFIE and the preconditioned Brakhage-Werner integral equation.

Theorem 20. *For every $k \notin F_D(\Omega^-)$, the operators $\widehat{L}^{-1}L$ of the preconditioned EFIE and the operator $\widehat{L}_{BW}^{-1}L_{BW}$ of the preconditioned Brakhage-Werner integral equation are similar in the sense that*

$$\left(L^{-1}L_{BW}\right)^{-1}\widehat{L}^{-1}L\left(L^{-1}L_{BW}\right) = \widehat{L}_{BW}^{-1}L_{BW}.$$

Proof. The proof begins with the equality given by lemma 19:

$$L\widehat{L}^{-1} = L_{BW}\widehat{L}_{BW}^{-1}.$$

Composing the above equation on the left by L_{BW}^{-1} implies that

$$\left(L^{-1}L_{BW}\right)^{-1}\widehat{L}^{-1} = \widehat{L}_{BW}^{-1}.$$

Now, the identity operator $I = LL^{-1}$ is introduced in the left hand side to obtain

$$\left(L^{-1}L_{BW}\right)^{-1}\widehat{L}^{-1}LL^{-1} = \widehat{L}_{BW}^{-1}.$$

Finally, it suffices to compose to the right by the invertible operator L_{BW} to derive the sought relation

$$\left(L^{-1}L_{BW}\right)^{-1}\widehat{L}^{-1}L\left(L^{-1}L_{BW}\right) = \widehat{L}_{BW}^{-1}L_{BW}.$$

□

Consequently, the operator of the preconditioned EFIE and the preconditioned Brakhage-Werner integral equation are similar. Due to theorem 13, this result can be extended to the preconditioned MFIE and preconditioned CFIE.

Remark 21. *So far, only the indirect integral equation of Brakhage-Werner has been studied. However, theorem 20 can be extended for any couple of boundary integral equations, provided that they are obtained by applying directly the Dirichlet boundary condition to the total field. This extended result can be proven in the same way as theorem 20 and summarized as follows for two general integral formulations. For $j = 1, 2$, let \mathcal{T}_j be a volume integral operator, such as the single-layer operator \mathcal{L} , the double-layer operator \mathcal{M} or any linear combination of them, even involving an invertible operator such as in [3]. For $j = 1, 2$, the total field of the integral formulation numbered by j is sought as $u_T = \mathcal{T}_j\varphi_j + u^{inc}$, where φ_j is the unknown density, solution of the boundary integral equation j :*

$$T_j\varphi_j = -\gamma_0^+ u^{inc}, \quad \text{with } T_j = \gamma_0^+ \mathcal{T}_j. \quad (42)$$

Hence, if the two integral equations (42) are uniquely solvable and equivalent to the scattering problem, then the following equality holds true

$$(T_1^{-1}T_2)^{-1}\widehat{T}_1^{-1}T_1(T_1^{-1}T_2) = \widehat{T}_2^{-1}T_2,$$

where, for $j = 1, 2$, \widehat{T}_j is the single scattering operator of T_j (defined in the same way as above). In other words, the two preconditioned boundary integral equations are similar.

5 Numerical results

Provided k is not an irregular frequency and after being preconditioned, the EFIE, MFIE, CFIE and the Brakhage-Werner integral equation are equal or similar for the last case. As a consequence, their operator have the same spectrum. Numerically, this also implies that the behavior of an iterative solver will be the same for every integral equation. The numerical results presented in this section aim to illustrate these properties. They also aim to show that the single scattering preconditioner accelerate the convergence rate of the iterative solver. In our examples, the boundary integral equation are discretized thanks to the boundary element method, which is briefly described in the first paragraph only for the EFIE, the method being similar for the other integral equations. The second paragraph is devoted to the numerical results.

5.1 Boundary element method: the example of the EFIE

Recall that the EFIE for the Dirichlet boundary value problem reads as

$$L\rho = -u^{inc}|_{\Gamma}.$$

Let Γ be described by a polygonal approximation Γ_h containing N_h segments. The largest length of the segments will be denoted by h . The finite element approximation space V_h of $L^2(\Gamma_h)$ contains all the continuous and piecewise linear functions on Γ_h (\mathbb{P}_1 finite elements):

$$V_h := \left\{ \rho_h \in \mathcal{C}^0(\Gamma_h) / \rho_h|_{K_j} \in \mathbb{P}_1, 1 \leq j \leq N_h \right\}.$$

The weak formulation of the EFIE is then given by

$$\begin{cases} \text{Find } \rho_h \in V_h \text{ such that} \\ \forall \Phi_h \in V_h, (L_h \rho_h, \Phi_h)_{L^2(\Gamma_h)} = - (u^{inc}|_{\Gamma_h}, \Phi_h)_{L^2(\Gamma_h)}. \end{cases}$$

After approximating the double integrals using quadrature formulæ, the weak formulation have then the following matrix form

$$\begin{cases} \text{Find } \boldsymbol{\rho}_h \in \mathbb{C}^{N_h} \text{ such that} \\ [L_h] \boldsymbol{\rho}_h = -[M_h] \mathbf{u}^{inc}, \end{cases}$$

where $[L_h] \in \mathcal{M}_{N_h, N_h}(\mathbb{C})$ is the matrix of the single-layer potential, $[M_h] \in \mathcal{M}_{N_h, N_h}(\mathbb{C})$ is the mass matrix for the linear finite elements, $\boldsymbol{\rho}_h \in \mathbb{C}^{N_h}$ the nodal vector of the density and $\mathbf{u}^{inc} \in \mathbb{C}^{N_h}$ the incident nodal vector. Then, the matrix of the single scattering $[\hat{L}_h]$ is computed by extracting and inverting the blocks located on the diagonal of the matrix $[L_h]$. Finally, the system approaching the preconditioned EFIE is given by

$$\begin{cases} \text{Find } \boldsymbol{\rho}_h \in \mathbb{C}^{N_h} \text{ such that} \\ [\hat{L}_h]^{-1} [L_h] \boldsymbol{\rho}_h = -[\hat{L}_h]^{-1} [M_h] \mathbf{u}^{inc}. \end{cases} \quad (43)$$

The method is exactly the same for the other integral equations. In the example that follows, the single scattering preconditioner is obtained using a direct method, in the sense that the matrix of the linear system is fully computed and numerically inverted using a direct solver. However, in practice and as highlighted in the Introduction, the size of the linear system could be too large to compute and invert the matrix (*e.g.* $[\hat{L}_h]$ in equation (43)), especially at high frequency and/or with a large number of obstacles. In that case, an iterative solver coupled with a fast and low cost matrix-vector product must be used, such as Fast Multipole Method. The

single scattering preconditioning could then be applied considering another iterative procedure. Nevertheless and for the sake of clarity, the preconditioner is here still denoted by $[\hat{L}_h]^{-1}$.

Lastly, computing the single scattering preconditioner involves the inverse of the single scattering matrix ($[\hat{L}_h]$ in the above example). Numerically, this operation is strongly dependent on the conditioning of the matrix and thus of the associated integral equation. Therefore, even if theorems 13 and 20 show that the preconditioned integral equations are exactly the same or similar for the four integral equations, a well conditioned integral equation is still needed.

5.2 Numerical example

Three different kinds of scatterers are considered: ellipsoidal, rectangular and “kite-shaped” (see Figure 1(a) for the last shape). A total of 30 obstacles, 10 from each different shape, are randomly distributed in the box $[0, 60] \times [0, 60]$ with a random characteristic size of order 1. The distance b_{pq} between the centers of two obstacles Ω_p^- and Ω_q^- is such that $b_{pq} \geq 3$. The direction of the incident plane wave is set to $\beta = (\cos(\pi/2), \sin(\pi/2))$ and the wavenumber k to 20. Finally, the mesh is generated using 15 points per wavelength, implying that every obstacle is represented by around 300 segments. An example of a geometry is shown on Figure 1(b). The parameters α and η of the CFIE are chosen with respect to what appears as a reasonable choice [14, section 5.2.1] (other choices would be more optimal as highlighted in remark 17):

$$\alpha = 0.2 \quad \text{and} \quad \eta = -ik = -20i.$$

Concerning the Brakhage-Werner integral equation and for simplicity, the parameter η_{BW} is set to the optimal choice in the case of single scattering by a unit disk (see remark 17 for references for other geometries):

$$\eta_{BW} = i\frac{k}{2} = 10i.$$

The four boundary integral equations EFIE (15), MFIE (16), CFIE (18) and Brakhage-Werner (32) are approximated using boundary element method of degree 1, leading to four linear systems and four matrices denoted respectively by A_{EFIE} , A_{MFIE} , A_{CFIE} and A_{BW} . Then, each linear system is preconditioned by its single scattering matrix, which is denoted by a hat symbol (*e.g.* \hat{A}_{EFIE} for A_{EFIE}). With these notation and according to theorems 13 and 20, the following equalities must hold true

$$\hat{A}_{EFIE}^{-1}A_{EFIE} = \hat{A}_{MFIE}^{-1}A_{MFIE} = \hat{A}_{CFIE}^{-1}A_{CFIE},$$

and

$$\hat{A}_{EFIE}^{-1}A_{EFIE} = (A_{EFIE}^{-1}A_{BW})\hat{A}_{BW}^{-1}A_{BW}(A_{EFIE}^{-1}A_{BW})^{-1}.$$

It appears that, indeed, these four matrices are numerically close to each other:

$$\frac{\|\hat{A}_{EFIE}^{-1}A_{EFIE} - \hat{A}_{MFIE}^{-1}A_{MFIE}\|_{\infty}}{\|\hat{A}_{EFIE}^{-1}\hat{A}_{EFIE}\|_{\infty}} \leq 4.10^{-2}, \quad \frac{\|\hat{A}_{MFIE}^{-1}A_{MFIE} - \hat{A}_{CFIE}^{-1}A_{CFIE}\|_{\infty}}{\|\hat{A}_{MFIE}^{-1}A_{MFIE}\|_{\infty}} \leq 4.10^{-2},$$

$$\frac{\|\hat{A}_{EFIE}^{-1}A_{EFIE} - \hat{A}_{CFIE}^{-1}A_{CFIE}\|_{\infty}}{\|\hat{A}_{CFIE}^{-1}A_{CFIE}\|_{\infty}} \leq 8.10^{-3},$$

and

$$\frac{\|\hat{A}_{EFIE}^{-1}A_{EFIE} - (A_{EFIE}^{-1}A_{BW})\hat{A}_{BW}^{-1}A_{BW}(A_{EFIE}^{-1}A_{BW})^{-1}\|_{\infty}}{\|\hat{A}_{BW}^{-1}A_{BW}\|_{\infty}} \leq 4.10^{-4}.$$

where $\|\cdot\|_\infty$ denotes the infinite matrix norm: $\|A\|_\infty = \max_i \sum_j |a_{i,j}|$ for any complex matrix $A = (a_{ij})$. Thus, these results illustrate theorems 13 and 20. From a practical point of view, these two propositions also imply that the matrices of the four preconditioned integral equations share the exact same spectrum. To observe this, Figures 1(c) and 1(d) present their eigenvalues in the complex plane. It appears that the four spectra coincide. More precisely, the relative error $|a - b|/|a|$ between two eigenvalues a and b reaches 2.7% at its maximum. Hence, theorems 13 and 20 seem to be numerically satisfied in the sense that the matrices of the preconditioned integral equations are very close to each other (up to a change of basis for the Brakhage-Werner integral equation).

To observe the effect of the preconditioner on an iterative solver, the eight linear systems (four integral equations and four preconditioned integral equations) are solved using GMRES [24] with a restart of 50 and a tolerance of 10^{-6} . The history of convergence, presented on Figures 1(e) and 1(f), show that, first, EFIE and MFIE do not converge without being preconditioned. Adding to this, it appears that the preconditioner accelerates the convergence of the GMRES. Indeed, the number of iterations involved by the GMRES is decreased by 36% and 23% for respectively the CFIE and Brakhage-Werner. This can be a consequence of the cluster of eigenvalues centered on the point $(1, 0)$, observed on Figures 1(c) and 1(d). However, this paper is not intended to study the impact of the preconditioning on the convergence rate and the choice of the parameters is discussable. Nevertheless, these examples show that, first, the four curves representing the four preconditioned integral equations are superimposed, and second that the preconditioner seems to accelerate and, at least, does not deteriorate the convergence rate.

6 Conclusion

This paper deals with boundary integral equation preconditioning for the multiple scattering problem. Two main results were established, which can be summarized as follows. After being preconditioned by their single scattering operator, firstly, every direct integral equations lead to the exact same equation, and secondly, the indirect integral equation of Brakhage-Werner becomes similar to the direct integral equations. In particular, applying the single scattering preconditioner to whichever integral formulation leads to the exact same convergence rate of the Krylov subspaces solver.

To conclude this article, two brief remarks can be done. First, the above results could probably be extended to the Maxwell's equations. Second, it should be kept in mind that the single scattering preconditioner involves the inversion of the M diagonal blocks of the boundary integral operator. Numerically, these last operations strongly depend on the considered integral formulation and moreover, at high frequency, they become costly and must be handled by iterative methods. On the other hand, the multiple scattering problem is also treated by an "outer" Krylov subspace solver. Hence, this will lead to launch M inner Krylov solvers at each iteration of the outer solver.

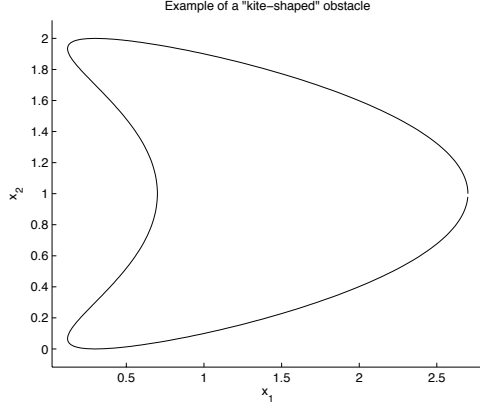
Acknowledgement

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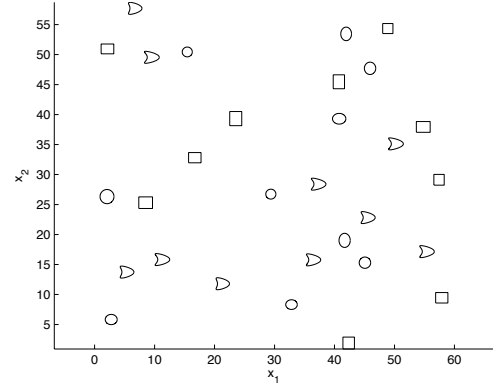
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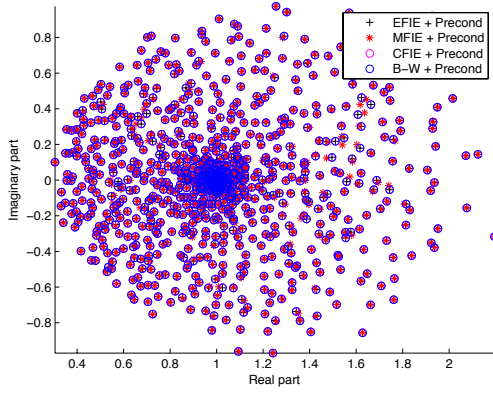
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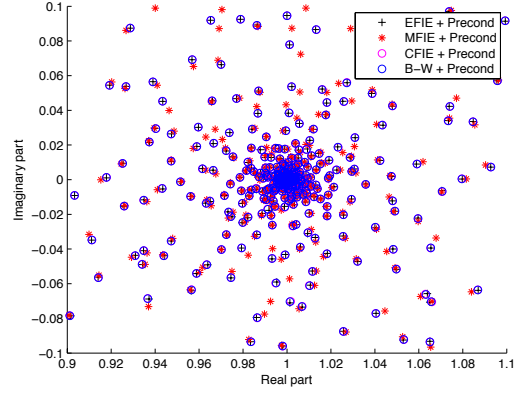
(a) Example of a “kite-shaped” obstacle



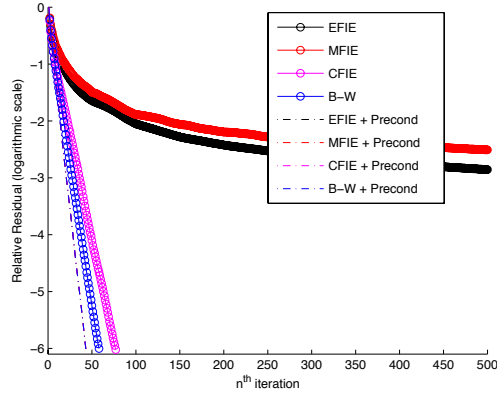
(b) Configuration



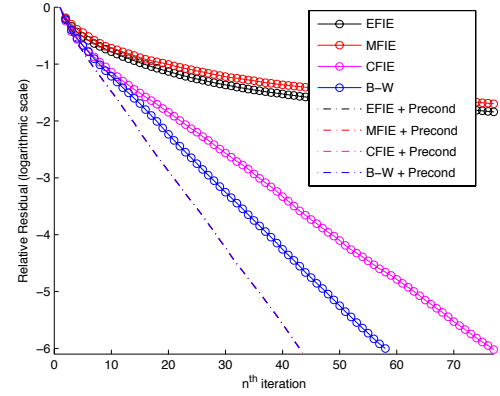
(c) Eigenvalues of the 4 preconditioned operators



(d) Zoom of Figure (c) around the point (1,0)



(e) History of convergence of the GMRES(50,10⁻⁶)



(f) Zoom of Figure (e) on the four curves of interest

Figure 1: The “kite-shaped” obstacle is presented on figure (a) and picture (b) presents the $M = 30$ obstacles with the three possible shapes: ellipses, rectangles and kites. The scatterers are randomly placed in the box $[0, 60]^2$, such that the distance b_{pq} between the centers of two obstacles Ω_p^- and Ω_q^- satisfies $b_{pq} \geq 3$. figure (c) shows the (numerical) eigenvalues of the four preconditioned operators and figure (d) is a zoom around the point (1,0). Finally, Figures (e) and (f) show the history of convergence of the GMRES(50,10⁻⁶) for the four integral equations and their preconditioned version (represented by “+ Precond”). It appears that the four curves of the preconditioned integral equations are superimposed.